1202 Summer 2011: Solutions

1 (a) A *group* consists of a set G together with a (closed) binary operation $\star$ such that

(α) $\star$ is associative;

(β) G has an identity element (w.r.t.$\star$);

(γ) each element of G has an inverse (w.r.t. $\star$)

Here a *(closed) binary operation* is a map $\star : G \times G \longrightarrow G$, where the image of (a, b) is denoted $a \star b$;

$\star$ is *associative* means that for all $a, b, c \in G$, $a \star (b \star c) = (a \star b) \star c$;

an *identity element* is an element $e \in G$ such that $e \star g = g \star e = g$ for all $g \in G$ (the identity element is in fact unique);

an *inverse* of an element $g \in G$ is an element $h \in G$ such that $g \star h = h \star g = e$, where e is the identity element.

(i) $G = \{f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\}$, $(f \star g)(x) = f(x)g(x)$. This is a group.

$\star$ is clearly a (closed) binary operation on G.

$\star$ is associative: $((f \star g) \star h)(x) = (f \star g)(x)h(x) = ((f(x)g(x))h(x)$ and $(f \star (g \star h)(x) = f(x)(g \star h)(x) = (f(x)(g(x)h(x))$ and these two last expressions are the same since multiplication in $\mathbb{R}$ is associative. Hence $((f \star g) \star h)(x) = (f \star (g \star h))(x)$ for all x and so $(f \star g) \star h = f \star (g \star h)$.

Let e be the function defined by $e(x) = x$ for all $x \in \mathbb{R}_+$. Then $e \in G$ and $(f \star e)(x) = f(x)e(x) = f(x)1 = f(x) = (e \star f)(x)$ for all x, so $f \star e = f = e \star f$. Thus e is the identity element.

Let $f \in G$. Define a function $g \in G$ by $g(x) = 1/f(x)$. This is well-defined since $f(x) > 0$, so $1/f(x)$ is defined and positive. Easy check that $f \star g = e = g \star f$.

(ii) $G = \{f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\}$, $(f \star g)(x) = f(g(x))$. This fails to be a group.

The identity function id defined by $id(x) = x$ for all x is clearly the identity element. However, not all elements of G have inverses: for example, let $f \in G$ be defined by $f(x) = 1$ for all $x \in \mathbb{R}_+$. Then if g were the inverse of f, then $g \star f = id$, so $(f \star g)(2) = id(2) = 2$, i.e. $1 = 2$.

(iii) $G = \{f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+\}$, $(f \star g)(x) = |f(x) - g(x)|$. This fails to be a group.

$\star$ is not associative: take functions f, g, h, so that $f(1) = 1$, $g(1) = 2$, $h(1) = 4$. Then $(f \star g)(1) = |f(1) - g(1)| = |1 - 2| = 1$, so $(f \star g) \star h))(1) = |1 - 4| = 3$. $(g \star h)(1) = |2 - 4| = 2$, so $(f \star (g \star h))(1) = |1 - 2| = 1$. Thus $((f \star g) \star h))(1) \neq (f \star (g \star h))(1)$ and hence $(f \star g) \star h) \neq f \star (g \star h)$.

2. (a) **Stage 1 Definition of cosets.**
For any $g \in G$, the coset $Hg = \{hg : h \in H\}$
Stage 2 G is the union of the cosets.
Since $e \in H$, for any $g \in G$, $g = eg \in Hg$. Hence $G = \cup_{g\in G} Hg$.
Stage 3 Cosets are either the same or don't intersect.
We prove that for any two cosets Hg and Hg', **either** $Hg = Hg'$ **or** $Hg \cap Hg' = \emptyset$.
So suppose that $Hg \cap Hg' \neq \emptyset$, say $x \in Hg \cap Hg'$. Then $x = h_1 g = h_2 g'$ for some $h_1, h_2 \in H$. Hence $g = h_1^{-1}h_2 g'$ and so for any $h \in H$, $hg = hh_1^{-1}h_2 g'$. Now since H is a subgroup, $hh_1^{-1}h_2 \in H$ and hence $hg \in HG'$. Thus we have proved that $Hg \subseteq Hg'$. Similarly $Hg' \subseteq Hg$ and hence $Hg = Hg'$.
Stage 4 G is the disjoint union of some of the cosets.
We know that $G = \cup_{g\in G} Hg$, and that any two cosets are either equal or have empty intersection. Therefore, if we leave out the repetitions, we get a *disjoint* union

$$G = Hg_1 \cup Hg_2 \cup \ldots \cup Hg_r$$

for some suitable choice of elements $g_1, g_2, \ldots, g_r \in G$.
Stage 5 All cosets are the same size.
We show that for any $g \in G$, $|Hg| = |H|$. So define $\phi : H \longrightarrow Hg$ by $\phi(h) = hg$. By the definition of the coset Hg, ϕ is surjective. If $\phi(h) = \phi(h')$, then $hg = h'g$ and hence $hgg^{-1} = h'gg^{-1}$, i.e. $h = h'$. Hence ϕ is also injective, and so bijective. It follows that $|Hg| = |H|$.
Stage 6 The result.
Since $G = Hg_1 \cup Hg_2 \cup \ldots \cup Hg_r$ is disjoint, we have $|G| = |Hg_1| + |Hg_2| + \ldots + |Hg_r|$. Then using Stage 5, we have that $|G| = r|H|$ and hence $|H|$ divides $|G|$.

(b) Let $x, y \in H \cap K$. Then $x \in H$ and $y \in H$: since H is a subgroup, $x^{-1}y \in H$. Similarly $x^{-1}y \in K$. Hence $x^{-1}y \in H \cap K$. Thus $x, y \in H \cap K$ implies $x^{-1}y \in H \cap K$. Also since H is a subgroup $e \in H$: since K is a subgroup $e \in K$: hence $e \in H \cap K$. Thus $H \cap K$ is a subgroup of G.

$H \cap K$ is clearly also a subgroup of H: hence by Lagrange's Theorem, $|H \cap K|$ divides $|H| = 5$. Similarly $|H \cap K|$ divides $|K| = 7$. Hence $|H \cap K| = 1$, i.e. $H \cap K = \{e\}$.

Let $X = \{hk : h \in H, k \in K\}$. X is a subset of G. All the elements of X are distinct, for suppose $h_1k_1 = h_2k_2$ $(h_i \in H, k_i \in K)$. Then $h_2^{-1}h_1 = k_2k_1^{-1}$. Since H is a subgroup, $h_2^{-1}h_1 \in H$, and since K is a subgroup $k_2k_1^{-1} \in K$. Hence $h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K = \{e\}$, and so $h_2^{-1}h_1 = k_2k_1^{-1} = e$, i.e. $h_2 = h_1$, $k_2 = k_1$. Thus X consists of 35 distinct elements, and since $|G| = 35$, $X = G$. Thus every element of G can be written uniquely as hk for some $h \in H, k \in K$.

3. (a) $\det A = \sum_{\sigma \in S_n} sgn(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{n,\sigma(n)}$

(b) Write $B = A^T$, so $b_{ij} = a_{ji}$. Then $\det(A^T) = \det(B)$
$= \sum_{\sigma \in S_n} sgn(\sigma) b_{1,\sigma(1)} b_{2,\sigma(2)} \dots b_{n,\sigma(n)} = \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1),1} \dots a_{\sigma(n),n}$
Now write $\mu = \sigma^{-1}$. As σ ranges over S_n, so does $\mu = \sigma^{-1}$, so
$\det(A^T) = \sum_{\mu \in S_n} sign(\mu^{-1}) a_{\mu^{-1}(1),1} \dots a_{\mu^{-1}(n),n} = \sum_{\mu \in S_n} sign(\mu) a_{\mu^{-1}(1),1} \dots a_{\mu^{-1}(n),n}$
(since $sgn(\mu^{-1}) = sgn(\mu)$).
Now $a_{\mu^{-1}(1),1} \dots a_{\mu^{-1}(n),n} = \prod_{j=1}^{n} a_{\mu^{-1}(j),j}$).
As j varies between 1 and n, so does $\mu^{-1}(j)$: so writing $k = \mu^{-1}(j)$,
$\prod_{j=1}^{n} a_{\mu^{-1}(j),j} = \prod_{k=1}^{n} a_{k,\mu(k)}$.
Hence
$\det(A^T) = \sum_{\mu \in S_n} sgn(\mu) \prod_{k=1}^{n} a_{k,\mu(k)} = \sum_{\mu \in S_n} sgn(\mu) a_{1,\mu(1)} a_{2,\mu(2)} \dots a_{n,\mu(n)} = \det(A)$.

(c) $b_{ij} = \mathbf{v}_i.\mathbf{v}_j$. Define A to be the 3×3 real matrix with columns $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, so $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$. Then $A^T = \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{pmatrix}$ and hence $A^T A$ has (i,j)-entry $(\mathbf{v}_i)^T \mathbf{v}_j = \mathbf{v}_i.\mathbf{v}_j = b_{ij}$.
Thus $B = A^T A$ and hence $\det B = \det A^T \det A = (\det A)^2 \geq 0$

(d) $$\det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 2 & 7 & 5 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \end{pmatrix}$$
$$= \det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 3 \end{pmatrix} = -\det \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 3 \end{pmatrix} =$$
$$= \det \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -2.$$

d

4. (i) $A = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}$.

$c_A(t) = \det \begin{pmatrix} t-2 & -3 \\ -4 & t+2 \end{pmatrix} = t^2 - 16 = (t-4)(t+4)$. Hence eigenvalues of A are 4 and -4.

$\lambda = 4$: then eigenvector is solution to $\begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$, e.g. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Similarly $\lambda = -4$ yields eigenvector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Hence if we take $P = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$, then $P^{-1}AP = D = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$ is diagonal.

8 marks

(ii) First note $P^{-1} = \frac{1}{8}\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$.

Now $A^n = (PDP^{-1})^n = PD^nP^{-1} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & (-4)^n \end{pmatrix}(1/8)\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} =$
$(1/8)4^n \begin{pmatrix} 3 & -(-1)^n \\ 2 & 2(-1)^n \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = (1/8)4^n \begin{pmatrix} 6+2(-1)^n & 3-3(-1)^n \\ 4-4(-1)^n & 2+6(-1)^n \end{pmatrix}$

8 marks

(iii) $X^2 = A$ if and only if $(P^{-1}XP)^2 = P^{-1}AP = D$.

There are four obvious solutions Y_1, Y_2, Y_3, Y_4 to $Y^2 = D = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$, namely $\begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 2i \end{pmatrix}$. This yields four solutions, X_1, X_2, X_3, X_4, to $X^2 = A$, namely $X = P\begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 2i \end{pmatrix}P^{-1} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 2i \end{pmatrix}(1/8)\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$
$= \frac{1}{8}\begin{pmatrix} \pm 6 & \mp 2i \\ \pm 4 & \pm 4i \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \pm 6 \pm 2i & \pm 3 \mp 3i \\ \pm 4 \mp 4i & \pm 2 \pm 6i \end{pmatrix}$

Explicitly, solutions are:

$X_1 = \frac{1}{4}\begin{pmatrix} 6+2i & 3-3i \\ 4-4i & 2+6i \end{pmatrix}$

$X_2 = \frac{1}{4}\begin{pmatrix} -6+2i & -3-3i \\ -4-4i & -2+6i \end{pmatrix}$

$X_3 = \frac{1}{4}\begin{pmatrix} 6-2i & 3+3i \\ 4+4i & 2-6i \end{pmatrix}$

$X_4 = \frac{1}{4}\begin{pmatrix} -6-2i & -3+3i \\ -4+4i & -2-6i \end{pmatrix}$

Could there be any more solutions? If $X^2 = A$, then if $Y = P^{-1}XP$, $Y^2 = D$. So suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$. Then

(i) $a^2 + bc = 4$

(ii) $ab + bd = 0$

(iii) $ca + dc = 0$

(iv) $cb + d^2 = -4$.

If $a + d = 0$, then $4 = a^2 + bc = d^2 + bc = -4$, a contradiction, so from (ii) and (iii) $b = 0$ and $c = 0$. Hence $a^2 = 4$ and $d^2 = -4$ and this yields exactly the four solutions Y_1, Y_2, Y_3, Y_4 already found. Hence X_1, X_2, X_3 and X_4 are the only solutions to $X^2 = A$.

5. (a) (i) The *eigenspace* $E_{\lambda_i} = \{\mathbf{v} \in \mathbb{C}^n : A\mathbf{v} = \lambda_i \mathbf{v}\}$.
(ii) The *geometric multiplicity* e_i of λ_i is $\dim(E_{\lambda_i})$.
(iii) The *characteristic polynomial* $c_A(t)$ of A is given by $\det(tI - A)$.
(iv) Write $c_A(t) = (t-\lambda_1)^{f_1}...(t-\lambda_r)^{f_r}$. Then the *algebraic multiplicity* of λ_i is f_i.

(b) Suppose A has n distinct eigenvalues, say λ_i $(i = 1, 2, ..., n)$, with corresponding eigenvectors $\mathbf{v}_i$. We claim $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is LI. Suppose not, then take a relation of dependence involving as few as possible terms. By renumbering, we can assume we have a relation

$\sum_{i=1}^{r} \alpha_i \mathbf{v}_i = \mathbf{0}$ (1)

where all $\alpha_i \neq 0$. Now multiplying (1) by A we get

$A\sum_{i=1}^{r} \alpha_i \mathbf{v}_i = \mathbf{0}$, so

$\sum_{i=1}^{r} \alpha_i A\mathbf{v}_i = \mathbf{0}$, so

$\sum_{i=1}^{r} \alpha_i \lambda_i \mathbf{v}_i = \mathbf{0}$ (2)

Now taking (2) from $\lambda_r \times$ (1) we get

$\sum_{i=1}^{r-1} \alpha_i(\lambda_i - \lambda_r)\mathbf{v}_i = \mathbf{0}$ (3)

Since all the λ_i are different, and all the α_i are non-zero, this is a relation involving $r-1$ of the terms, i.e. shorter than (1), a contradiction.
Hence the set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is LI and hence forms a basis for $\mathbb{R}^n$; by the basic criterion, A is diagonalisable.

(c) A is diagonalisable if and only if $e_i = f_i$ $(i = 1, ..., r)$.

$$A = \begin{pmatrix} 1-b & b \\ 1-a-b & a+b \end{pmatrix}$$

$c_A(t) = (t-1+b)(t-a-b) - b(1-a-b) = t^2 - (a+1)t + a = (t-1)(t-a)$.
Hence eigenvalues are a and 1.
Case 1 $a \neq 1$. Then A has two distinct eigenvalues and hence is diagonalisable.
Case 2 $a = 1$. Then $c_A(t) = (t-1)^2$ and so A has just one eigenvalue 1 with $f_i = 2$.

$A = \begin{pmatrix} 1-b & b \\ -b & 1+b \end{pmatrix}$ and so eigenspace is

$$E_1 = \{\begin{pmatrix} x \\ y \end{pmatrix} : A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}\}$$
$$= \{\begin{pmatrix} x \\ y \end{pmatrix} : (A - I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$
$$= \{\begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} -b & b \\ -b & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}.$$

Clearly E_1 is 2-dimensional (so $e_1 = 2$) if and only if $b = 0$.
Thus A is diagonalisable if and only if ($a \neq 1$ or $a = 1$ and $b = 0$).

6. (a) Let λ be an eigenvalue of A (in $\mathbf{C}$) with corresponding eigenvector $\mathbf{v}$ (in $\mathbf{C}^n$), so

$$A\mathbf{v} = \lambda\mathbf{v}$$

Taking the complex conjugate and transposing, we get

$$\bar{\mathbf{v}}^T A = \bar{\lambda}\mathbf{v}^T$$

Now we have

$$\bar{\lambda}\bar{\mathbf{v}}^T\mathbf{v} = \bar{\mathbf{v}}^T A\mathbf{v} = \bar{\mathbf{v}}^T\lambda\mathbf{v} = \lambda\bar{\mathbf{v}}^T\mathbf{v}$$

so

$$(\bar{\lambda} - \lambda)\mathbf{v}^T\mathbf{v} = 0$$

Write $\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}^T$, where $v_j = a_j + ib_j$ $(a_j, b_j \in \mathbb{R})$. Since $\mathbf{v} \neq \mathbf{0}$, at least one a_j or $b_j > 0$. Now

$$\bar{\mathbf{v}}^T\mathbf{v} = \sum_{j=1}^n v_j\bar{v}_j = \sum_{i=1}^n (a_j^2 + b_j^2) > 0$$

and hence $\bar{\mathbf{v}}^T\mathbf{v} \neq 0$. Hence

$$\lambda = \bar{\lambda}$$

i.e. $\lambda \in \mathbb{R}$.

(b) $A = \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}$. $c_A(t) = (t-5)(t-7) - 3 = t^2 - 12t + 32$ $= (t-4)(t-8)$. Hence eigenvalues are 4 and 8, and corresponding eigenvectors are $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$. These are already orthogonal, so we must just normalize them to get the orthogonal matrix

$P = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$.

Then $P^{-1}AP = P^TAP = D = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$.

(c) $5x^2 - 2\sqrt{3}xy + 7y^2 = 4$. Write $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 5 & -\sqrt{3} \\ -\sqrt{3} & 7 \end{pmatrix}$.

Equation can be written as $\mathbf{v}^TA\mathbf{v} = 4$. Let $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ and let $\mathbf{v} = P\mathbf{u}$. Then in new co-ordinates equation is $\mathbf{u}^TP^TAP\mathbf{u} = 4$, i.e. $\mathbf{u}^TD\mathbf{u} = 4$. Writing this in co-ordinates gives $u^2 + 2v^2 = 1$.

This is an ellipse and graph in (u, v)-plane shown in Diagram 1 below. To sketch this on (x, y)-plane, note that $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to $\begin{pmatrix} x \\ y \end{pmatrix} =$

$\begin{pmatrix}\sqrt{3}/2\\1/2\end{pmatrix}$ and $\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ corresponds to $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1/2\\\sqrt{3}/2\end{pmatrix}$; see Diagram 2.

Diagram 1

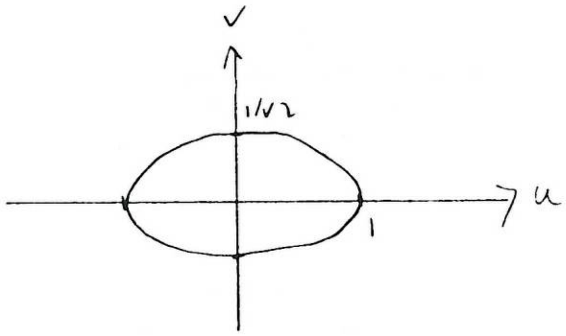

Diagram 2